DIRECTIONS OF ARRIVAL ESTIMATION

by

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DIRECTIONS OF ARRIVAL ESTIMATION

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CERTIFICATE

It is certified that the work contained in the thesis titled 'DIRECTIONS OF ARRIVAL ESTIMATION' , by *M Ramulu* has been carried out under my supervision and that this work has not been submitted elsewhere for a degree

May, 1995

Dr Govind Sharma
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DEDICATED

TO

MY BROTHERS

ACKNOWLEDGEMENTS

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Life has been very enjoyable especially in the company of Prakash C Ramesh Ranganatham Kalee CCS Reddy Maruthi NK Reddy Mahesh PS Subramanium S Subramanium Verma Naidu Surendra Srikanth and all other 1993 EE batch mates

I never forget the help rendered by my friends outside this campus May my bond to each one of you be unaffected even after we diverge to different walks of life

I have always cherished the kind of moral and spiritual support my family members have provided me Words are not sufficient to express my faith respect and love for them If I do any thing worthwhile most of the credit should go to them

Ramulu

ABSTRACT

An attempt has been made to find out the Directions Of Arrivals (DOA) of narrow band signals (which are non coherent) received over a uniform linear array The DOA algorithms like MUltiple SIgnal Classification (MUSIC) are critically dependent on computation of noise eigenvectors The MUSIC algorithm involves any of the matrix iterative procedures of the complex Hermitian matrix (auto correlation matrix of the process) is bound to have the limitations posed by the particular matrix iterative procedure both in terms of computational time and complexity For fast real time applications alternatively Direct Noise Subspace Basis (DNSB) method has been studied for real data DNSB method is simple and computationally efficient Unlike the above methods the Subspace Based Approach which directly uses information from both the noise and signal subspaces has been verified Comparisions with the above three methods are made For one source case the CR bound has been derived and verified

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Chapter 1

Introduction

The practical problems of interest in array signal processing are extracting the desired parameters such as the directions of arrival (DOA) power levels and cross correlations of the signals present in the scene from the available information including the measured data Often one may also be specifically interested in the actual signal of one of these sources and in this case it is necessary to estimate the actual waveform associated with the desired signal by improving the overall reception in an environment having several sources To achieve this ideally it should be possible to suppress the undesired signals and enhance the desired signal

Sensor arrays have been in use for several decades in many practical signal processing applications Such an array consists of a set of sensors that are spatially distributed at known locations with reference to a common reference point These sensors collect signals from sources in their field of view Depending on the sensor characteristics and the path of propagation the source wavefronts undergo deterministic and/or random modifications The sensor outputs are composed of these source components and additive noise such as measurement and thermal noise [1]

The most common means for distinguishing a desired signal from an interference signal is the existence of a known frequency band within which the desired signal must fall We have to know the desired signal well enough so that it can be distinguished from interference signals it is reasonably to expect that the nature of the desired signal is known even though certain signal parameters (such as DOA, amplitude and phase) may have to be estimated

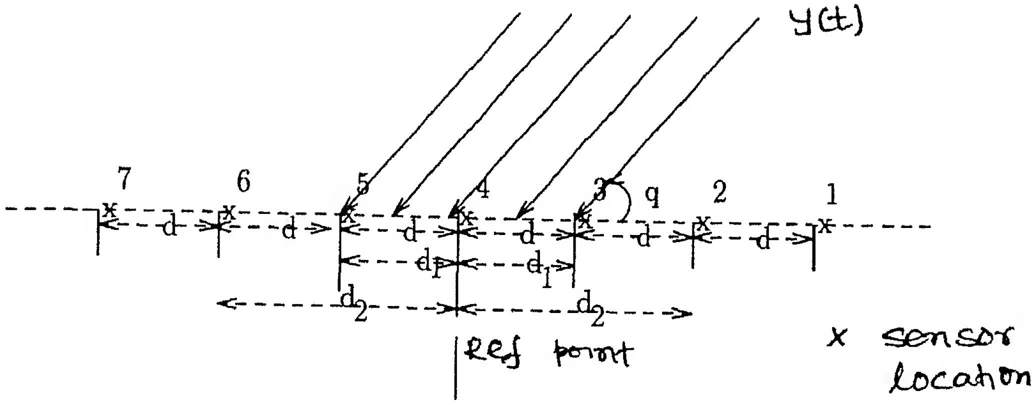

Figure 1 1 Linearly spaced array of sensors

1 1 Problem Formulation

Signal is modelled as a sample function of a random process Assumptions are made that the transmission medium is homogeneous so that the signal wavefront is perfetly coherent over the entire array(i e the signal components received by various sensors are delayed replicas of each other) The noise is considered additive white gaussian (AWG) and incoherent from sensor to sensor

In the fig 1 1 the plane wavefront is making an angle θ on the array broad side with the line joining sensors in the linear array, d is the distance between the sensors and it is normalised with respect to $\frac{\lambda}{2}$, the half wavelength associated with narrow band signal $y(t)$ Also d_1 d_2 are similarly normalised distances of the sensors from the reference point

The problem of Directions of arrival(DOA) estimation is solved and analyzed using various al gorithms like MUSIC(MUltiple SIgnal Clasification) method, Direct Noise Subspace Basis (DNSB) method, and Subspace based approach to parameter estimation(here, Directions of arrival)

1 2 Organization Of Thesis

Chapter 2 contains a concise but complete in itself description of the important Directions Of Arrival (DOA) estimation algorithms

Chapter 3 deals with the case of single source in detail. It gives the theriotical lower bound on variance of the bias of the parameter $\omega = \pi \cos\theta$ which when infinite snapshots are taken then the CR bound depends on the SNR and number of sensors. Here we assume that our estimator is unbiased.

Chapter 4 gives the simulation results followed by a breif discussion.

Chapter 2

Dirctions Of Arrival Estimation

The complex envelop of the total received signal at i^{th} sensor is given by [2]

$$x_i(t) = y(t)exp((-j\pi d \ \cos\theta)) + n \ (t) \quad (2\ 1)$$

$n_i(t)$ is the noise at the i^{th} sensor and

$$E[n_i(t)n_j(t)] = \sigma^2\delta_{ij} \quad (2\ 2)$$

The generalisation of signal model at the i^{th} sensor for M identical sensors receiving signals from K narrow band signals $y_1(t)$ $y_2(t)$, $y_K(t)$ that arrive at the array from directions θ_1 θ_2 θ_K with respect to line of array is given by

$$x_i(t) = \sum_{=1}^{K} y_r(t)exp(-j(\pi d \ \cos\theta_r + 2\pi f_r t)) + n \ (t) \quad (2\ 3)$$

Where f_r's are the input sinusoidal frequencies Dropping the sinusoidal t dependence in the summation we have

$$x_i(t) = \sum_{r=1}^{K} y_r(t)exp(-j\pi d \ \cos\theta_r) + n_i(t) \quad (2\ 4)$$

The array output vector $X(t)$, therefore can be written as

$$X(t) = AY(t) + n(t) \quad (2\ 5)$$

where $Y(t) = [y_1(t)\ y_2(t) \quad y_K(t)]^t$ (signal vector)
$n(t) = [n_1(t)\ n_2(t) \quad n_M(t)]^t$ (noise vector)
$A = \sqrt{M}[a_{\omega_1}\ a_{\omega} \quad a_{\omega_I}]$ is called the direction vector matrix of size $M \times K$ where $\omega_k = \pi \cos\theta_k$
and
$a(\omega_k)$ = direction vector (of size M) is given by

$$a(\omega_k) = \frac{1}{\sqrt{M}}[exp(-j d_1 \omega_k) \qquad exp(-j d_M \omega_k)]^t \tag{2 6}$$

$\frac{1}{\sqrt{M}}$ is used for normalisation of the direction vector Under the assumption that the signals and noises are stationary zeromean uncorrelated random processes the array output covariance matrix is given by

$$S = E[X(t) X^H(t)] = ARA^H + \sigma^2 I \tag{2 7}$$

R is the source covariance matrix of size $K \times K$

In the following section we will briefly describe some of the important algorithms that exist for the solution of the Direction of arrival(DOA) problem We can classify them as Eigen strucure based methods and non eigen structure based methods MUSIC PISARENKO ESPIRIT GEESE etc comes under former one where as Bartlett s Beamformer method Capon s Minimum variance estimator and Linear prediction method comes under latter case First we will describe the latter case

2 1 Non Eigen Structure Based Methods

2 1 1 Beamformer

Beamforming ordinarily involves forming multiple beams from multielement arrays through the use of appropriate delay and weighting matrices The conventional beamformer introduces apprpriate delays at the output of the each sensor to compensate the propagation delays of the wavefront reaching the array [3] As the name implies, the array output weights are chosen to be phase factors required to steer the array along some specific direction θ i e

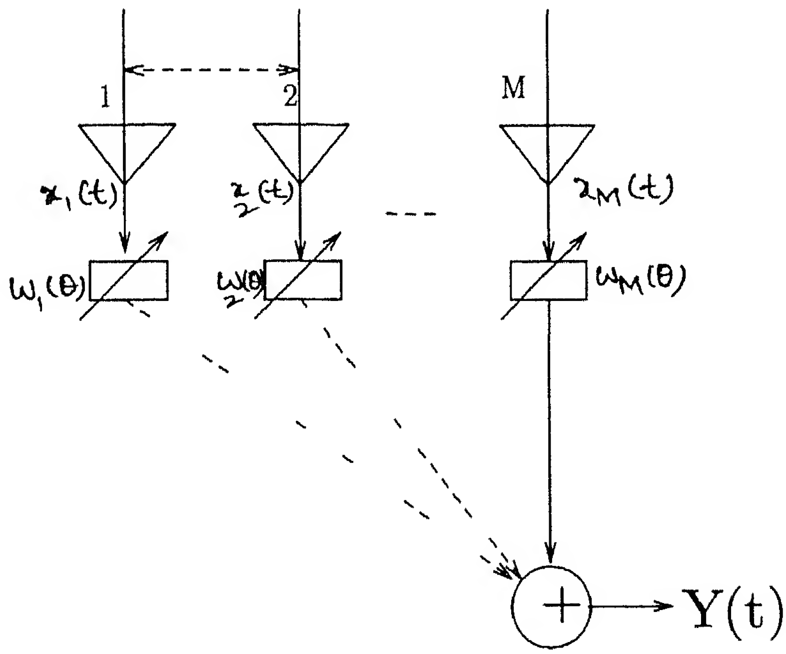

Figure 2 1 Beamforming

$$w_i = \frac{1}{\sqrt{M}} e^{(-j\pi d \ \cos\theta)} \tag{2 8}$$

For notational convenience define $\omega = \pi \cos\theta$

so that

$$W_B = \frac{1}{\sqrt{M}} [e^{(-jd_1\omega)}, e^{(-jd_2\omega)} \qquad e^{(-jd_M\omega)}]^t \triangleq a_\omega \tag{2 9}$$

Thus the array output

$$Y(t) = {W_B}^H X(t) \tag{2 10}$$

and the average output power

$$P_B(\omega) = E[|y(t)|]^2 = a^H(\omega) R a(\omega) \tag{2 11}$$

Where $R = E[X(t)X^H(t)]$

In a single target seen this estimator measures the actual power when steered along the true direction of arrival resulting a single peak in that direction. But it can be easily verified that if two sources are located inside the main beam the peaks in fact merge in to a single peak resulting in a loss of resolution.

2.1.2 Capon's Minimum Variance Estimator

In general the array output power contains power from the desired signal along the look direction as well as power in undesired ones along other direction of arrival. To minimize the contributions of this later set, the array output power is minimized while keeping the gain along the look direction constant [4]

$$\min W^H R W \text{ subject to } |w^H a(\omega)| = 1 \tag{2.12}$$

The solution to the weight vector for the positive covariance matrices is

$$W_c = \frac{R^{-1} a(\omega)}{a^H(\omega) R^{-1} a(\omega)} \tag{2.13}$$

substituting in eqn (2.11) for $a(\omega)$

$$P_C(\omega) = \frac{1}{a^H(\omega) R^{-1} a(\omega)} \tag{2.14}$$

The estimator in eqn (2.14) has superior resolving power compared to the Bartlett beamformer output.

2.1.3 Linear Prediction Method

In this case one of the sensor output is predicted as a linear combination of the remaining $(M-1)$ sensor outputs at any instant, and the predictor coefficients are selected so as to minimize the mean square error.

Letting $x_n, x_{n-1}, \ldots, x_{n-M+1}$ stand for the predictor for the M sensor outputs and x_n the predictor for x_n, we have

$$r_i = -\sum_{=1}^{M-1} a\, \mathrm{x}_{n-i} \tag{2 15}$$

This gives the error as

$$\mathcal{E}_n = x_n - r_n = -\sum_{=0}^{M-1} a\, x_{n-} \qquad a_0 = 1 \tag{2 16}$$

Minimization of the mean square error $E|\mathcal{E}_i|^2$ with respect to the unknowns results in the following standard set of linear equations

$$E[e_i\, x_k] = \sum_{i=0}^{M-1} a\ E[x_{n-i}\, x_{\ -k}] = 0\ k = 1\ 2 \qquad M-1 \tag{2 17}$$

and the final mean square error is given by $\mathcal{E}|e_i|^2 = \delta_{M-1}$ Under the assumption that the cross correlation between the sensor outputs (the sensor outputs are spatially wide sense stationarity) depend only upon inter element distances then

$$E[x_i\ \ x_{n-k}] = r(k-i) = r^*(i-k) \tag{2 18}$$

and the eqn (2 19) becomes

$$\sum_{i=0}^{M-1} a_i\, r(k-i) = 0\ k = 1, 2, \qquad M-1 \tag{2 19}$$

and also

$$\sum_{i=0}^{M-1} a\ r(-i) = \delta_{M-1} \tag{2 20}$$

Thus X_n can be thought of as the output of a linear shift invariant system driven by an uncorrelated noise process with averege power δ_{M-1} as shown below

If the errors e_n e_{i+k} are also uncorrelated for all n k then the input represents a white noise process and x_n an autoregressive process of order $M-1$ From fig 2 2 the power spectral density

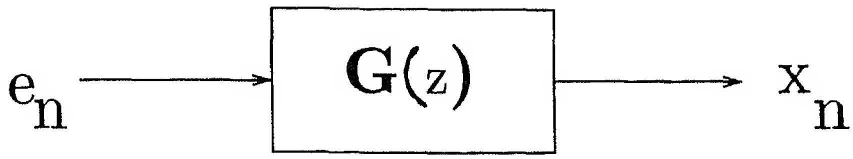

Figure 2 2 Linear prediction model

$S_e(\omega) = \delta_{M-1}$ through the relation

$$S_x(\omega) = |G(exp(j\omega)|^2 S_e(\omega) = \frac{\delta_{M-1}}{|H(exp(j\omega))|^2} \quad (2\ 21)$$

and

$$H(z) = H(exp(j\omega)) = 1 + a_1 z^{-1} + \cdots + a_{M-1} z^{-(M-1)} \quad (2\ 22)$$

The above polynomial has its zeros close to the unit circle and gives sharp peaks in the output spectrum correspond to the DOA estimate of the signals present The main disadvantage of the linear predictor is that it exhibits spurious peaks in the output spectrum whenever the array pointing direction is away from one of the actual direction of arrival [2]

2 2 Eigen Structure Based Methods

2 2 1 MUSIC

In the fig 2 3 shown below, there are M identical sensors and receiving signals from K narrow band signals $y_1(t)$ $y_2(t)$, $\ldots$ $y_K(t)$ that arrive at the array from directions $\theta_1, \theta_2, \ldots$ θ_K wrt the line of array

We assume here that none of the signals are coherent i e , $|\rho_{ij}| \neq 1$ Using superposition of signal contributions of all the incident planewaves, the received signal $x_i(t)$ at the i^{th} sensor can be written as

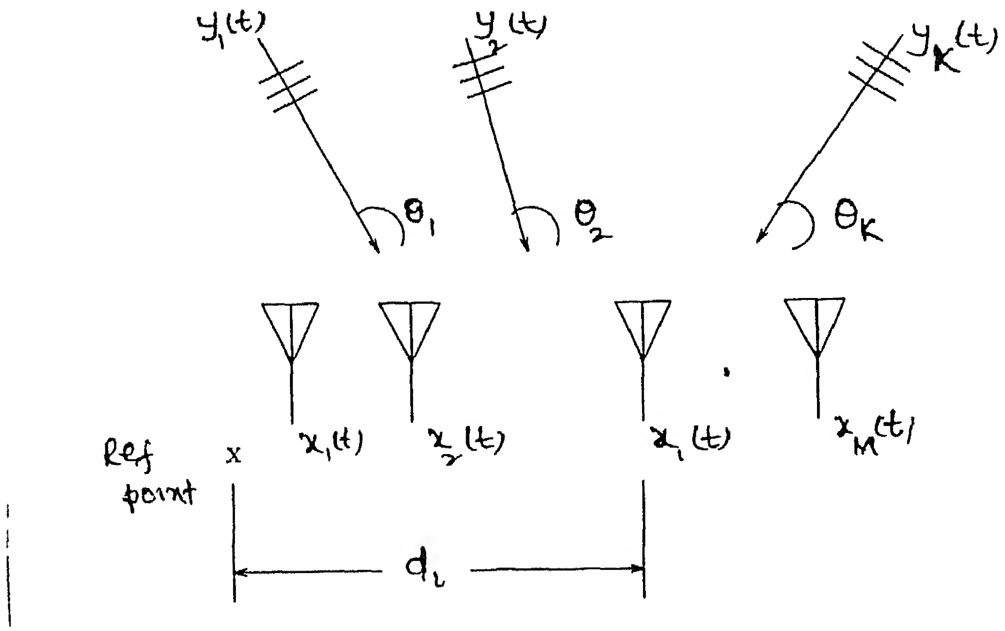

Figure 2 3 A typical array scene

$$x_i(t) = \sum_{k=1}^{K} y_k(t) e^{-j\pi d \ \cos\theta_k} + n_i(t) \tag{2 23}$$

where d_i is the i^{th} sensor element position, normalized distance to half of the common wave length, wrt some common reference point, and $n_i(t)$ stands for the additive noise at the i^{th} sensor It is assumed here that signals and noises are stationary zero mean, uncorrelated random processes, and further

$$E[n_i(t) n_j^H(t)] = \sigma^2 \delta_{ij} \tag{2 24}$$

From the eqn (2 5) the array output vector $X(t) = AY(t) + n(t)$

where $Y(t) = [y_1(t), y_2(t), \quad , y_K(t)]^t$

$n(t) = [n_1(t), n_2(t), \quad , n_M(t)]^t$ and

$$A = \begin{bmatrix} 1 & & 1 \\ e^{-j\omega_1} & & e^{-j\omega_K} \\ & & \\ e^{-j(M-1)\omega_1} & & e^{-j(M-1)\omega_K} \end{bmatrix} \tag{2 25}$$

is an $(M \times K)$ matrix consisting of K direction vectors From our assumptions $M \times M$ array output covariance matrix $S = E[X(t)X^H(t)]$ has the form

$$S = AE[Y(t)Y^H(t)] + E[n(t)n^H(t)] = ARA^H + \sigma^2 I \qquad (2\ 26)$$

Here we see that covariance matrix is hermitian and positive definite and source covariance matrix $R = E[Y(t)Y^H(t)]$ of size $K \times K$ which is non singular when there are no coherent sources are present in the data

Assuming $M > K$ and that A is full rank K which implies that nonnegative definite matrix ARA^H is also of rank K Therefore S will have $M - K$ repeated lowest eigenvalues $(=\sigma^2)$

Therefore if V is an eigenvector of S for the eigenvalue σ^2 we have

$$ARA^H V = 0 \qquad (2\ 27)$$

Unitary diagonalizability of S implies orthonormal decomposition of the data space in to noise and signal subspaces of dimension $M - K$ and K respectively
Eqn (2 27) can be further simplified to

$$A^H V = 0 \qquad (2\ 28)$$

Stated in words the eigenvectors associated with the repeating lowest eigenvalue of S are orthogonal to the direction vectors corresponding to the actual arrival of angle

However owing to the presence of noise in the eigenvector estimates that span these two sample subspaces(signal and noise subspaces) the orthogonality relations of eqn (2 28) are no longer valid The best that we can do is to search for the signal vectors that are most closely orthogonal to the noise subspace Accordingly in the MUSIC algorithm it is proposed to estimate the angular frequevcies of the complex sinusoids in the input signal as the peaks of the following sample

spectrum

$$S(\omega) = \frac{1}{\sum_{=K+1}^{M} |a^H(\omega)V|^2} \tag{2 29}$$

Here we see that for the above equation to be meaningful $M \geq K+1$ and this restricts the minimum number of required sensor elements to atleast one more than the total number of sources present in the scene

In this chapter we briefly explain Direct Noise Subspace Basis method and Sub Space based approach to estimate the directions of arrival from S the correlation matrix which we shall assume has been fairly well calculated (by possibly fair sample mean of data snap shot covariances)

2 3 Direct Noise Subspace Basis Vectors Finding

$ARA^HV = 0$ if and only if $A^HV = 0$

Now we shall develop an algorithm to compute V directly from the correlation matrix which determines noise sub space Since the zeros of (2 28) give the DOA estimates it is necessary to find all the $M - K$ vectors V satisfying (2 28)

To simplify derivations we shall assume a uniform end element referenced and seperation normalised array so that $d_i = i-1, i = 1\ 2, \quad , M$ Thus $(A_{n\ i}) = e^{-jd_m\omega_n}$ $m = 1 \quad M$ $n = 1 \quad K$ and therefore, $(A_{m\ n}) = e^{-jd_n\omega_n}$ $m = 1 \quad K$ $n = 1 \quad M$ Also let P be the right shift operator on R^M so that P is a matrix is

$$P = \begin{bmatrix} 0 & 0 & & 0 \\ 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ & & & \\ 0 & & 1 & 0 \end{bmatrix} \tag{2 30}$$

and $P^M = 0$

2 3 1 Theorem 1

Let $M \geq 2K$ And $A^t = [1|A_0||B_0]$ with $A_0 = K \times K$ and $B_0 = (K \times (M - K - 1))$ be a partition of A Then the set v Pv $P^{M-K-1}v$ where $v = [1|(A_0^{-1}1)|0 \quad 0]^t$ forms a set of basis for noise subspace of R

For the proof of this theorem please refer [5]

Given the above general form of the subspace basis vectors we next attempt to calculate these from the data Since these are the basis vectors of of $S_0 = S - \sigma^2 I$ we get from (2 26) that,

$$S_0 V = (S - \sigma^2 I)V = ARA^H V = 0 \tag{2 31}$$

Let us partition S_0 as follows

$$\begin{bmatrix} S_1 & S_3 \\ S_2 & S_4 \end{bmatrix} \begin{bmatrix} 1 \\ v_1 \\ v_k \\ \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \tag{2 32}$$

where S_1 is $(K+1) \times (K+1)$ and other S_i 's are compatible This gives

$$\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \begin{bmatrix} 1 \\ v_1 \\ \\ v_k \end{bmatrix} = 0 \tag{2 33}$$

$$rank \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \leq K \tag{2 34}$$

Again let s be the first column of

$$\begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$$

and use the partition

$$[s|S^-]\begin{bmatrix}1\\v_1\\ \\v_k\end{bmatrix}=0 \tag{2 35}$$

to give

$$S^-\begin{bmatrix}1\\v_1\\ \\v_k\end{bmatrix}=-s \tag{2 36}$$

Here it can be seen that rank $S^- \leq K$ (the equality holds always if $M \leq 2K - 1$) In the present case if S^- is full rank v is directly determined once eqn (2 36) is solved by using the left inverse of S^-

The rank of S^- may not satisfy the above eqn(2 34) in case of the data in real environment Then by using methods like Gausian elimination or similar method of rank testing we can obtain good results

2 4 Subspace Based Approach To Parameter Estimation

This approach [6] starts with a model of second order data(Correlation matrix) generated from the samples of M received signals $x_i(n)$ $i = 1\ 2 \quad M$ Each of these signals contains either a known or estimated number K of moderately correlated signals $y_k(n)$ $k = 1\ 2 \quad K$ having distinct direction of arrivals(DOAS) $\theta_1\ \theta_2 \quad \theta_K$ with respect to a linear array of point sensors embedded in additive independent and identically distributed(i i d) Gaussian noise

The dimension of the signal subspace is estimated by comparing the magnitudes of the eigen values of the correlation matrix and using a threshold

$$x_i(n) = \sum_{k=1}^{K} y_k(n)\exp(-j\omega_k(i-1)) + v_i(n) \quad i = 1\ 2 \quad M \tag{2 37}$$

The above model provided in (2 37)with respect to the direction of arrival(DOA) problem applies also in several other situations including the problem of parameter estimation of sinusoids

2 4 1 Analysis of the standard model

The standard model for the $(M \times M)$ correlation matrix S was given in (2 26)

$$S = E[x(n)x^{H}(n)] = ARA^{H} + \sigma^{2}I \tag{2 38}$$

Since S is Hermitian and is modeled as in (2 38) it must be diagonalizable by a unitary matrix B Therefore

$$\begin{aligned} B^{H}SB &= B^{H}((ARA^{H}) + \sigma^{2}I)B \\ &= Diag[(\mu_1 + \sigma^2) \quad (\mu_K + \sigma^2)\sigma^2 \quad \sigma^2] \end{aligned} \tag{2 39}$$

is a $(M \times M)$ diagonal matrix For notational brevity we deifne below the $(K \times K)$ and $((M - K) \times (M - K))$ diagonal matrices Δ_1 and Δ_2, respectively

$$\Delta_1 \triangleq Diag[\lambda_1 \lambda_2 \quad \lambda_K]\ \lambda = \mu_i + \sigma_2 \ \ for \ \ i = 1 \quad K \tag{2 40}$$

$$\Delta_2 \triangleq Diag[\lambda_{K+1} \quad \lambda_M]\ \lambda_{K+} = \sigma_2 \ \ for \ \ i = 1 \quad M - K \tag{2 41}$$

Denote the $(M \times K)$ matrix composed of the eigen vectors associated with the eigenvalues $\lambda_1\ \lambda_2 \quad \lambda_K$ by B_s (signal subspace eigenvector matrix) and the$(M \times (M - K))$ matrix composed of the $(M - K)$ eigenvectors associated with the eigenvalues $\lambda_i\ i = K + 1\ K + 2 \quad M$ by B_n (noise subspace eigenvector matrix) so that

$$B = [B_s\, B_n] \tag{2 42}$$

Now from (2 42) and (2 39) since $B^H B = I$

$$[B_s\ B_n]^H(ARA^H)[B_s\ B_n] + \sigma_2 I = \text{Diag}[\Delta_1\ \Delta_2] \tag{2.43}$$

The above equation leads to the following two equalities

$$\begin{aligned} ARA^H B_s &= B_s \Delta_1 - B_s (\sigma_2 I) \\ &= B_s \text{Diag}[\mu_1 \ \ldots \ \mu_K] \end{aligned} \tag{2.44}$$

$$ARA^H B_n = B_n \Delta_2 - B_n(\sigma_2 I) = 0 \tag{2.45}$$

Please note that in eqns (2.44) and (2.45). The I's denote identity matrices of orders K and $(M-K)$ respectively. Since AR is of rank K it follows from (2.45) that the $(M \times (M-K))$ matrix $A^H B_n$ must satisfy

$$A^H B_n = 0 \tag{2.46}$$

Since $B_s^H B_s = I$ eqn (2.44) implies $(B_s^H A)R(A^H B_s) = (Diag)[\mu_1\, \mu_2 \ \ldots \ \mu_K]$ which in turn implies that the $(K \times K)$ matrix $A^H B_s$ is nonsingular. Thus it follows from (2.44) that

$$\begin{aligned} A &= B_s \text{Diag}[\mu_1 \ \ldots \ \mu_K](A^H B_s)^{-1} R^{-1} \\ &= B_s C \end{aligned} \tag{2.47}$$

where $C = \text{Diag}[\mu_1 \ \ldots \ \mu_K](A^H B_s)^{-1} R^{-1}$ is nonsingular. It can be seen in the following section that the columns of matrix C are related to the eigenvectors of the matrix whose eigenvalues etimates the parameters(in this case DOA'S) in the standard model of the signal correlation matrix.

2.4.2 Derivation of the Algorithm

After defining,

$$a(z) = [1\ z \ \ldots \ z^{M-1}]^t \tag{2.48}$$

it is clear from (2 46) that $z_i = e^{j\omega_i}$ $i = 1\ 2$ K are the Mroots of the $(M-1)$th degree polynomial equations

$$[a(z)]^t b = b_0 + b_1 z + \quad + b_{M-1} z^{M-1} = 0 \tag{2 49}$$

where $b = [b_0\ b_1 \quad b_{M-1}]^t$ is any one of the columns of B_n Therefore

$$A^H b = 0 \tag{2 50}$$

Each column of B_i is associated with an equation similar to the eqn (2 50) The set of these $2(M-K)$ equations can be written as
$[a(z)]^t [b_1 \quad b_{2(M-K)}] = 0$

and from (2 50)

$$A^H [b_1 \quad b_{2(M-K)}] = 0 \tag{2 51}$$

Here it is assumed that $b_{M-1} \neq 0$ in (2 49) for if $b_{M-1} = 0$ the monomial of highest degree is selected Multiply (2 49) by z dividing through out by b_{M-1} and rearrange as follows

$$z^M + \frac{b_{M-2}}{b_{M-1}} z^{M-1} + \quad + \frac{b_0}{b_{M-1}} z = 0 \tag{2 52}$$

After associating a companion matrix D of order M with the polynomial on the left hand side of (2 52)

$$D = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ & & & & \\ 0 & 0 & 0 & & 1 \\ 0 & \frac{-b_0}{b_{M-1}} & \frac{-b_1}{b_{M-1}} & & \frac{-b_{M-2}}{b_{M-1}} \end{bmatrix} \tag{2 53}$$

it is clear that the polynomial equation in (2 52) may be written as

$\det[zI - D] = 0$ The desired roots $z = e^{j\omega}$ $i = 1, 2, \ldots, K$ on the unit circle are also the K unit magnitude eigenvalues λ_i $i = 1, 2, \ldots, K$ of D

2.4.3 Theorem 1

The $(K \times K)$ matrix $B_s^t D B$ is nonsingular whose eigen values uniquely estimate the K distinct parameters $e^{j\omega}$ $i = 1, 2, \ldots, K$ in the standard model of the signal correlation matrix

For the proof please refer [6]

Chapter 3

Evaluation Of C-R Bound For One Source Case

These bounds apply to unbiased estimators. They are useful in that the best possible performance (smallest variance) for an unbiased estimator may be determined [8]. Estimators whose variance is close or equals the bound can then said to be an optimal one.

3.1 Joint Likelyhood Function

The model for direction of arrival problem as we have shown in eqn (2.3) is

$$r_i(t) = \sum_{r=1}^{K} y_r(t) exp(-j(w_r t + (i-1)\omega_r)) + n_i(t) \tag{3.1}$$

For one source case let us define

$$\omega = \omega_1 = \pi \cos\theta_1 \tag{3.2}$$

We know from chapter 2 of Van trees [7] that if

$$x(t) = y(t, A) + n(t) \tag{3.3}$$

where $n(t)$ is zero mean white Gaussian Noise with power spectral density $\frac{N_0}{2}$, $x(t)$ is the observed signal, $y(t, A)$ is the transmitted waveform and A is the unknown parameter which has to be

estimated Then the log likelihood function is given by

$$\ln \Omega = \frac{2}{N_0}\int_0^T r(t)y(t\ A)dt - \frac{1}{N_0}\int_0^T y^2(t\ A)dt \tag{3 4}$$

To find the maximum likelihood estimate of A we have to find the value of A for which eqn(3 4) is maximum A necessary but not sufficient condition can be obtained by differentiating eqn (3 4) with respect to A and equating it to zero

$$\int_0^T [r(t) - y(t\ A)]\frac{\partial y(t\ A)}{\partial A} = 0 \tag{3 5}$$

Above equation provides us the values of A at which the derivative of the log likelihood function is zero

Now when we have more than one simultaneous observations as in our sensor array case we should find the Joint log likelihood function We can extend the eqn(3 4) directly to the case where our array has N sensors so that we have N observations Then

$$\frac{\partial \ln P_{x/a}(X/A)}{\partial A} = \frac{2}{N_0}\sum_{i=1}^{N}\int_0^T [x_i(t) - y\ (t\ A)]\frac{\partial y_i(t, A)}{\partial A}dt \tag{3 6}$$

where A is the unknown parameter and $x_i(t) = y\ (t\ \omega) + n\ (t)$ $\quad 0 < t < T$ where $n\ (t)$ is zero mean white Gaussian noise with p s d $\frac{N_0}{2}$

If A is a non random parameter then the Cramer Rao bound requires that

$$Var(\hat{A} - A) \geq -E[\frac{\partial^2 \ln P_{x/a}(X/A)}{\partial A^2}]^{-1} \tag{3 7}$$

Substituting A as ω Now

$$E[\frac{\partial^2 \ln P_{x/w}(X/W)}{\partial \omega^2}] = \frac{2}{N_0}\sum_{i=1}^{N} E\int_0^T [x_i(t) - y_i(t\ W)]\frac{\partial^2 y_i(t\ W)}{\partial \omega^2}dt - E[\int_0^T [\frac{\partial y_i(t\ w)}{\partial \omega}]^2 dt] \tag{3 8}$$

where we assume the derivatives exist In the first term

$$E[x_i(t) - y_i(t\ w)] = E[n\ (t)] = 0 \qquad (3\ 9)$$

In the second term there are no random quantities hence the expectation operation gives the integral itself
Substituting in eqn (3 7)

$$Var(\omega - \omega) \geq \frac{N_0}{2\sum_{i=1}^{N}\int_0^T[\frac{\partial y_i(t\ \omega)}{\partial\omega}]^2 dt} \qquad (3\ 10)$$

For the single source case

$$y_i(t\ \omega) = y_1 exp(-(jw_r t + (i-1)\omega)) \qquad (3\ 11)$$

which implies $\frac{\partial y_i(t\ \omega)}{\partial\omega} = -j(i-1)y_1 exp(-(jw_r t + (i-1)\omega))$
Substituting in the eqn (3 10)

$$Var(\omega - \omega) \geq \frac{N_0}{2\sum_{i=1}^{N}(i-1)^2 y_1{}^2 T} = \frac{3N_0}{y_1{}^2 TN(N-1)(2N-1)} \qquad (3\ 12)$$

or

$$Var(\omega - \omega) \geq \frac{6}{[TN(N-1)(2N-1)snr]} \qquad (3\ 13)$$

Here $T >> \frac{1}{2\omega}$

As expected, the CR bound for the frequency estimator is inversly proportional to snr ($\frac{y_1{}^2}{N_0/2}$) and N (number of sensors)

For single source case with $M = 3$ and $M = 5$ the performance of MUSIC estimator has been simulated for various snr ranges from 5 to 30dB

Chapter 4

Simulation Results

In MUSIC method it requires computation of perfect orthogonal eigenvectors correspond to minimum eigenvalue which has multiplicity of $M - K$. To compute the eigenvalues/vectors for a Hermitian matrix it requires practically infinite number of plane rotations and reflections. In principle iterations are stopped when the off diagonal elements are negligible to the working accuracy [9].

Jacobi iterative algorithm and Householder method produce almost exact orthogonal eigenvectors but may be inaccurate when the matrix under consideration has close eigenvalues [9][11]. Also Householder method is relatively faster than the latter one.

Hence no algorithm is best regarding the computation of eigen values/vectors of complex matrix[10].

The dimension of the signal subspace is estimated by comparing the eigenvalues of the correlation matrix and using a threshold. It can be seen from the plots shown in figs (4.1) and (4.2) that the two smaller eigen values (corresponding to noise subspace) are not same. Also from fig (4.2) we can say that Rank of the matrix $S_0 = S - \sigma_2 I$ is no longer equal to K and it is greater than K. The plots were shown for various snr's 10, 20, 30dB. And the angle of arrivals were at 30, 60deg.

As we have seen Direct Noise Subspace Basis (DNSB) method is simple and computationally efficient if we can estimate the sensor noise variance σ^2.

Simulations were carried out for one source and two sources with sensor sizes of $M = 3$ 4 and 5 In all the cases the data length was 100 and the input sinusoidal frequencies were 100 and 250 Hz for two source case and 100 Hz was for one source case Random numbers were taken for arbitrary seed value

The DNSB method performing well for $M = 3$ with $K = 1$ and $K = 2$ Even for $M = 4$ and 5 it helps to resolve the peaks at correct locations but spread is more relatively with MUSIC method This is because for $M > K + 1$ the condition on rank of $S^- < K$ is not achieved for real data environment

4 1 Single Target Case($K = 1$)

MUSIC method for k^{th} direction of arrival computes the function

$$P(\omega) = \frac{1}{\sum_{i=K+1}^{M} |a^H(\omega)_k V_i|^2} \tag{4 1}$$

will correspond to the true direction of arrival DNSB method resolves the peak by solving the equation

$$S_0 \begin{bmatrix} 1 \\ v_1 \\ 0 \end{bmatrix} = 0 \tag{4 2}$$

where $S_0 = (S - \sigma^2 I)$ which implies

$$S^-[v_1] = -s \tag{4 3}$$

where S^- is the second column of s_0 matrix Above equation is solved for v_1 by using left inverse $(S^-)^+ = (S^{-H} S^-)^{-1} S^{-H}$ of S^-

The noise eigenvectors V_1 V_2 using the right shift operator are

$$\begin{bmatrix} 1 \\ v_1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ v_1 \end{bmatrix}$$

respectively

Now by using eqn (4 1) we compute the power spectrum The plot for single sinusoid (arrival angle 60deg) is shown in fig (4 3) for $M = 3$ It can be seen that the spread is less in case of DNSB method than MUSIC method The snr was 20dB for the sinusoid

4 2 Double Target Case($K = 2$)

For $M = 3$

$$S^{-} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -s \tag{4 4}$$

where S^{-} is the $M \times K$ matrix consists of second and third columns of S_0 matrix

Due to the rank degeneracy in S^{-} matrix we have done partial pivoting before finding out Noise subspace basis vectors v_1 v_K We have searched for largest element in magnitude in the first column then by interchanging the rows we have put that element at (1 1) position Similarly second column is searched for the largest element in magnitude and bringing that element at the corresponding (2 2) position Hence so on and so fourth Also the diagonally decreasing order in magnitude is assured Finally we have forced the $M - K$ rows to zero

By using left inverse S^{-+} we compute v_1 v_2 then noise eigenvector V_1

Similarly for $M = 4$ and $M = 5$ we find out noise eigenvectors from that the eqn (4 1) to resolve the peaks

For $M = 3$and$K = 2$ MUSIC method fails to resolve the peaks at correct locations and the spread is more also Hence in this case DNSB method is the ideal choice Further more for $M = 4$ 5and$K = 2$ none of the two methods able to resolve the peaks Only one of the two angles could able to detect by them These plots are shown in figs (4 4), (4 5) and (4 6)

Angle distribution plots were shown in figs (4 7) to (4 12) The SNR was 20dB for each sinusoid

Clearly from these plots we can say that the DNSB is the best estimator for $M = 3$ and $K = 2$ as it has lower variance we took 100 iterations to plot these figs From the plots variance and bias were found using the statistical formulae The two arrival angles were at 30 and 60 degs But for $M = 5$ and $K = 2$ the same cannot conclude as the rank of the matrix $S_0 = (S - \sigma^2 I)$ is not equal to K

The two methods (MUSIC and DNSB) compared on the basis of comutational time In case of DNSB method the cpu time found out for the noise subspace basis $v_1 \quad v_K$ through Moore Penrose inverse from the S^- matrix And in MUSIC method it was to find out eigen values and eigen vectors from the correlation matrix S

For two source case with sensor sizes 5 7 and 10 cpu time found out for the two methods MUSIC takes 13 3ms, 24ms and 30ms, whereas DNSB takes 6 66ms 6 8ms and 7 5ms respectively

The CR bound as given by the eqn (3 13) is plotted assumming the estimator is unbiased so that the mean square error is equal to the variance The estimator nearly attains the CR bound above some threshold SNR (about 10dB for $M = 5$ and about 20dB for $M = 3$) Unfortunately below the threshold the mean squre error increases rapidly, which renders the estimator not useful

Finally from the plots shown in figs (4 13) and (4 14) the MUSIC algorithm gives acceptable performance for snr s above 10dB

Chapter 5

Conclusions And Suggestions For Further Work

We have made an attempt to estimate the DOA's using the Eigenstructure based methods MUSIC, Direct Noise Subspace Basis and Subspace Based Approach.

The signals are narrowband and therefore the amplitude and phase modulations are assumed constant as the wavefronts travel across the array. As signal travels from one sensor to another it undergoes a phase change. This phase change is a function of the angle of arrival. We have tried to estimate this angle of arrival with the above mentioned three methods in the presence of zero mean white gaussian noise with variance σ^2. All the three methods were verified with real data.

In chapter2 we have breifly described the three algorithms analytically. The DNSB method though it fails to perform that well for $M > K + 1$ as it was for $M = K + 1$ it is simple and computationally efficient.

In chapter3 analytical solution for for one source case has been derived. The estimate turned to be unbiased assuming high SNR.

The possible directions in which further study in this area can be done are:

For $M > K + 1$ one can find out the ways to make DNSB method to resolve the peaks with better resolution.

Analytical solution for CR bound for two source case with more than three sensors can also be derived to verify with simulation results.

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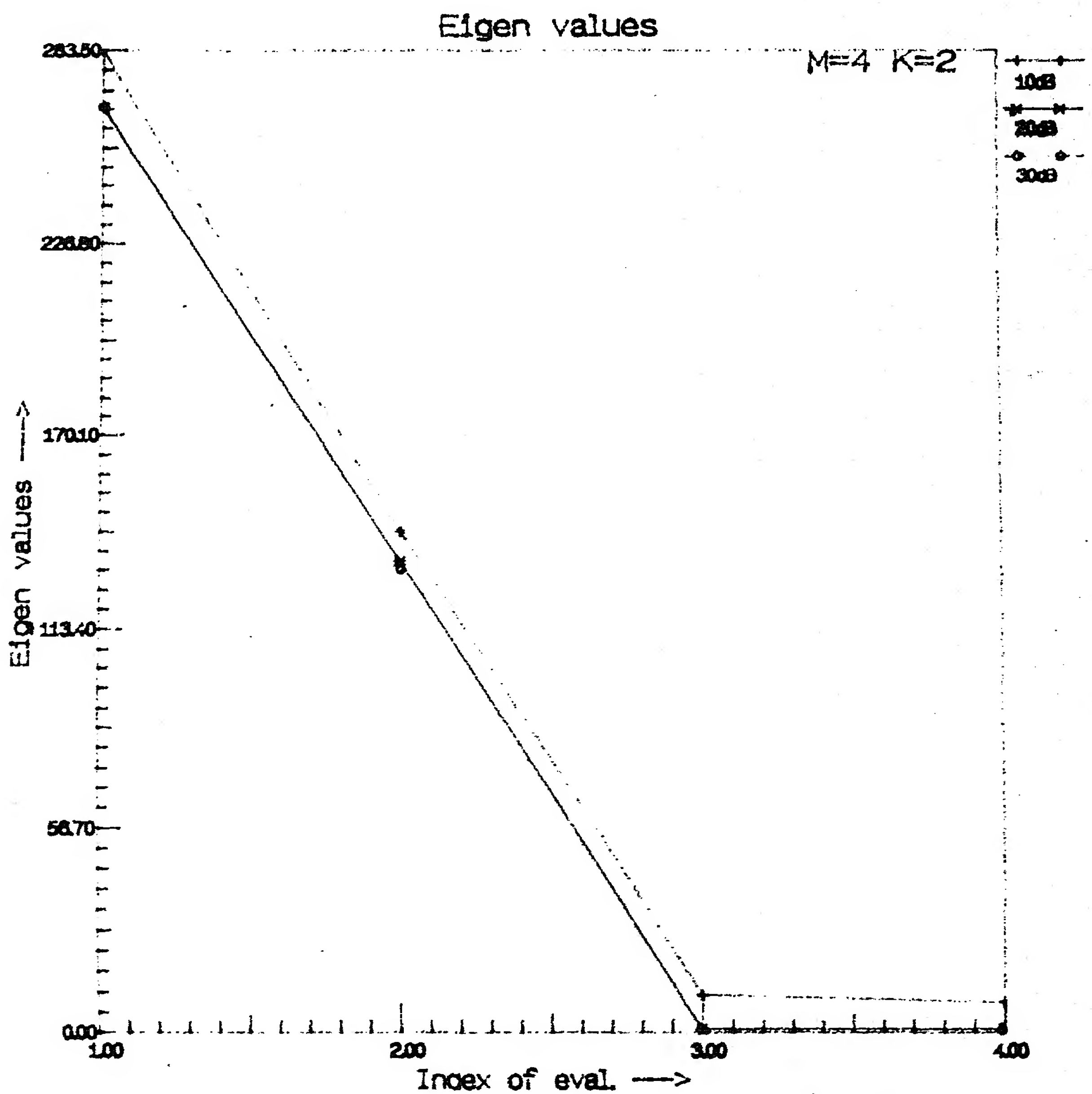


Eigen values
M=4 K=2
10dB
20dB
30dB
283.50
226.80
170.10
113.40
56.70
0.00
1.00
2.00
3.00
4.00
Eigen values --->
Index of eval. --->

Fig. 4.1

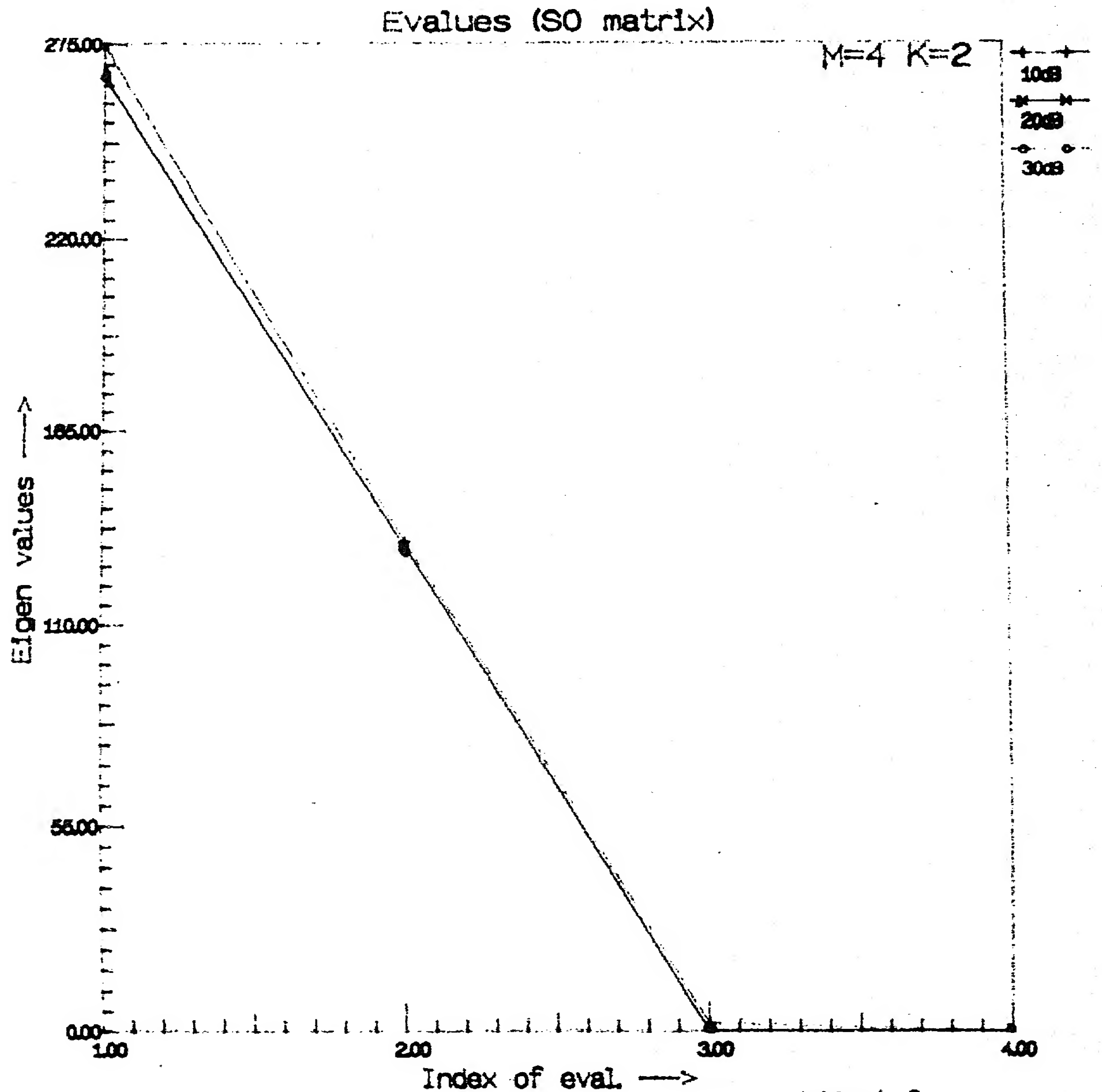

Evalues (SO matrix)
M=4 K=2
10dB
20dB
30dB
275.00
220.00
165.00
110.00
55.00
0.00
Eigen values ---->
1.00
2.00
3.00
4.00
Index of eval. --->

Fig. 4.2

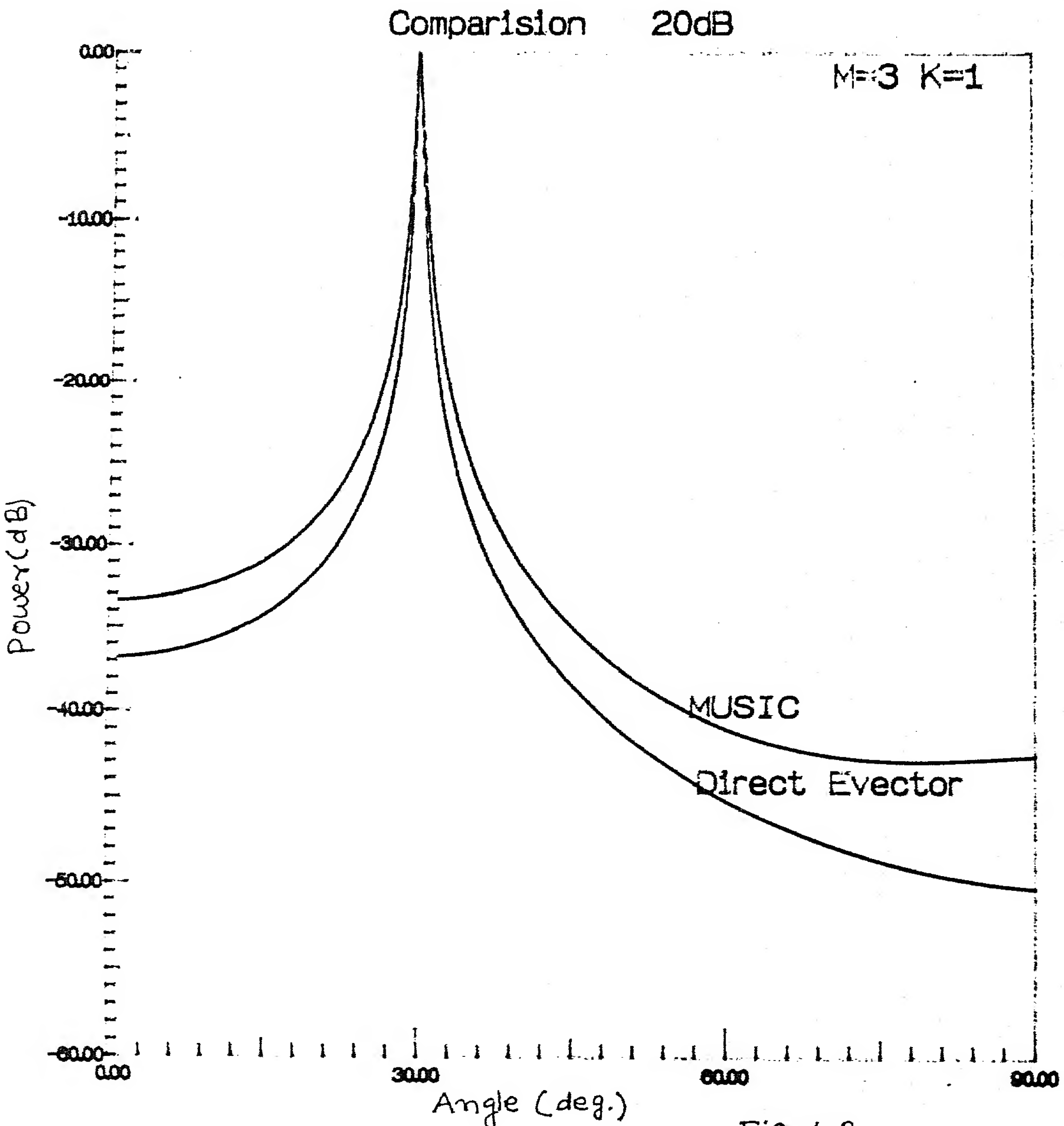
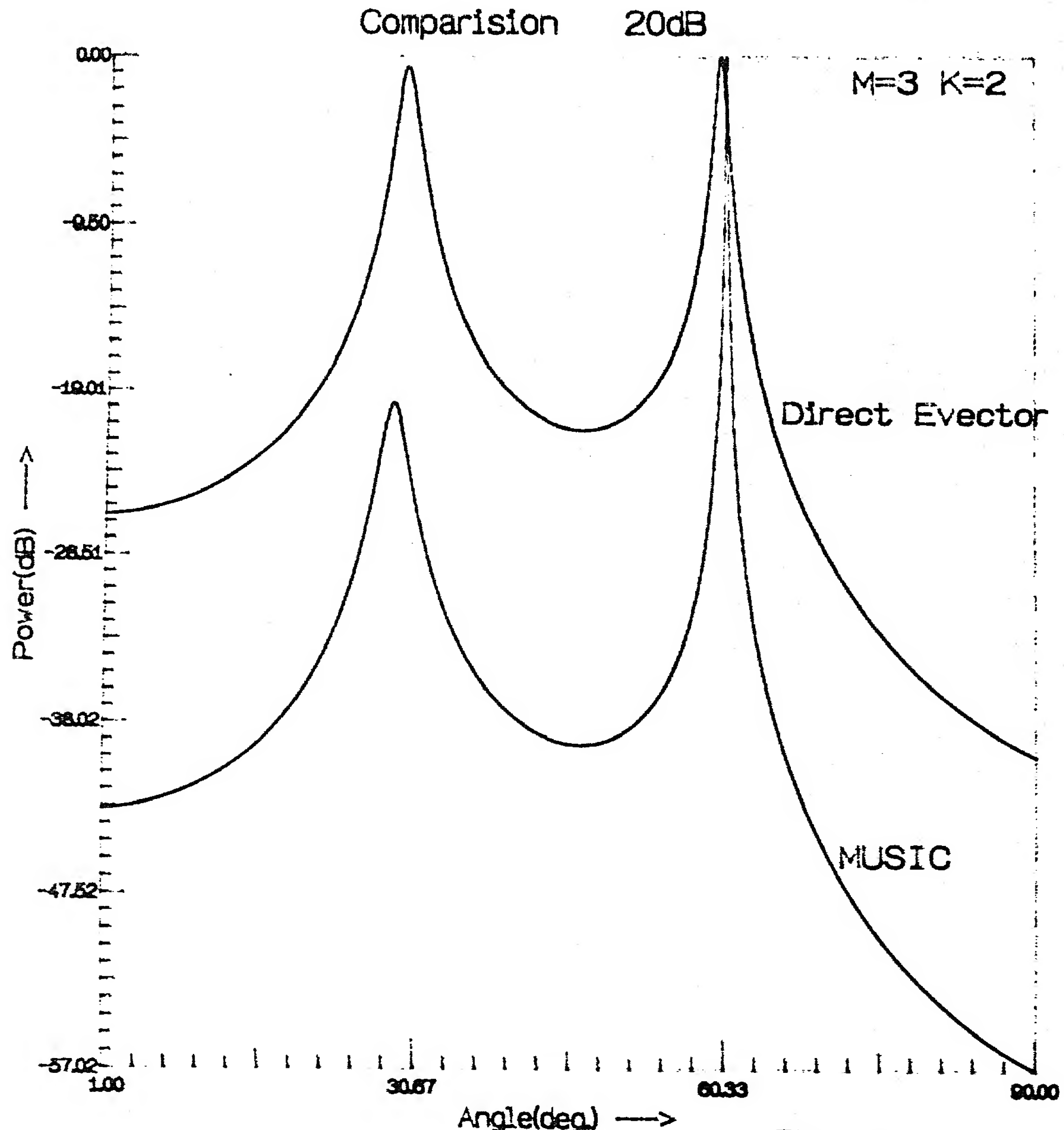
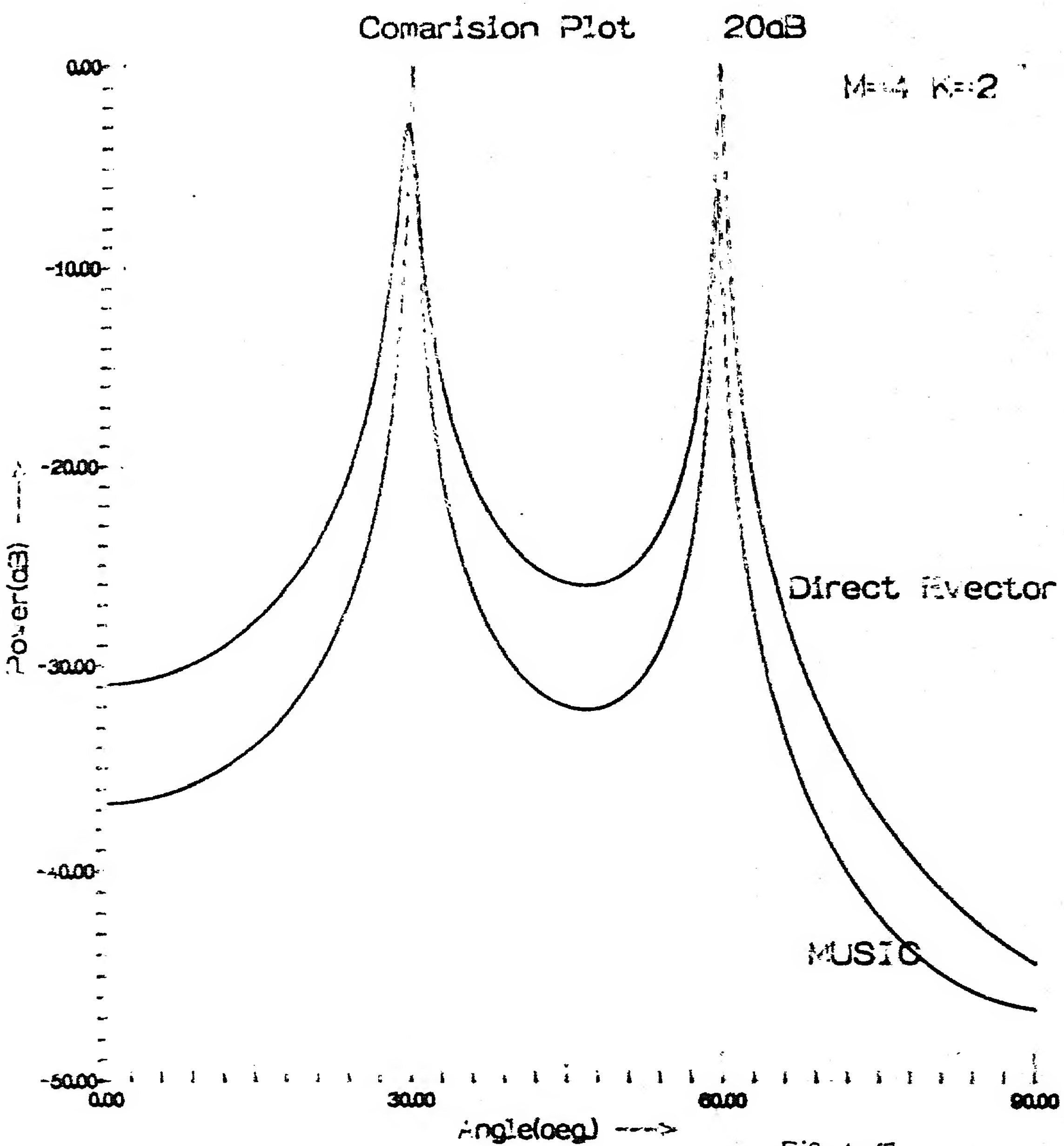

Comparision 20dB
M=3 K=1
MUSIC
Direct Evector
Power(dB)
0.00
-10.00
-20.00
-30.00
-40.00
-50.00
-60.00
0.00
30.00
60.00
90.00
Angle (deg.)

Fig. 4.3



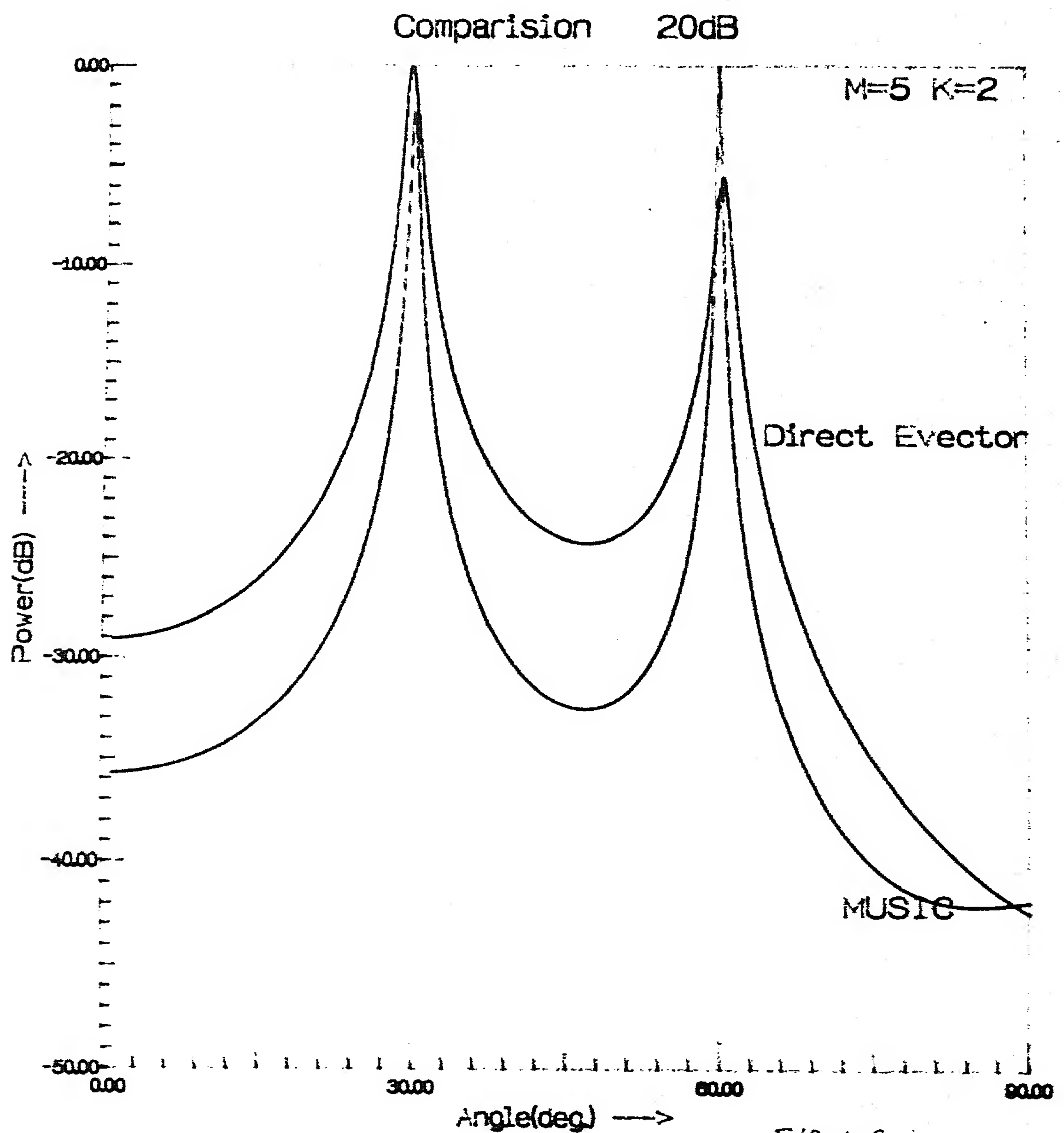
Comparision 20dB
M=3 K=2
Direct Evector
MUSIC
Power(dB) ---->
0.00
-9.50
-19.01
-28.51
-38.02
-47.52
-57.02
1.00
30.67
60.33
90.00
Angle(deg.) ---->

Fig. 4.4



Comarision Plot 20dB
M=4 K=2
Power(dB) --->
0.00
-10.00
-20.00
-30.00
-40.00
-50.00
0.00
30.00
60.00
90.00
Angle(deg) --->
Direct Evector
MUSIC

Fig. 4.5



Comparision 20dB
M=5 K=2
Direct Evector
MUSIC
0.00
-10.00
-20.00
-30.00
-40.00
-50.00
Power(dB) ---->
0.00
30.00
60.00
90.00
Angle(deg) --->

Fig. 4.6

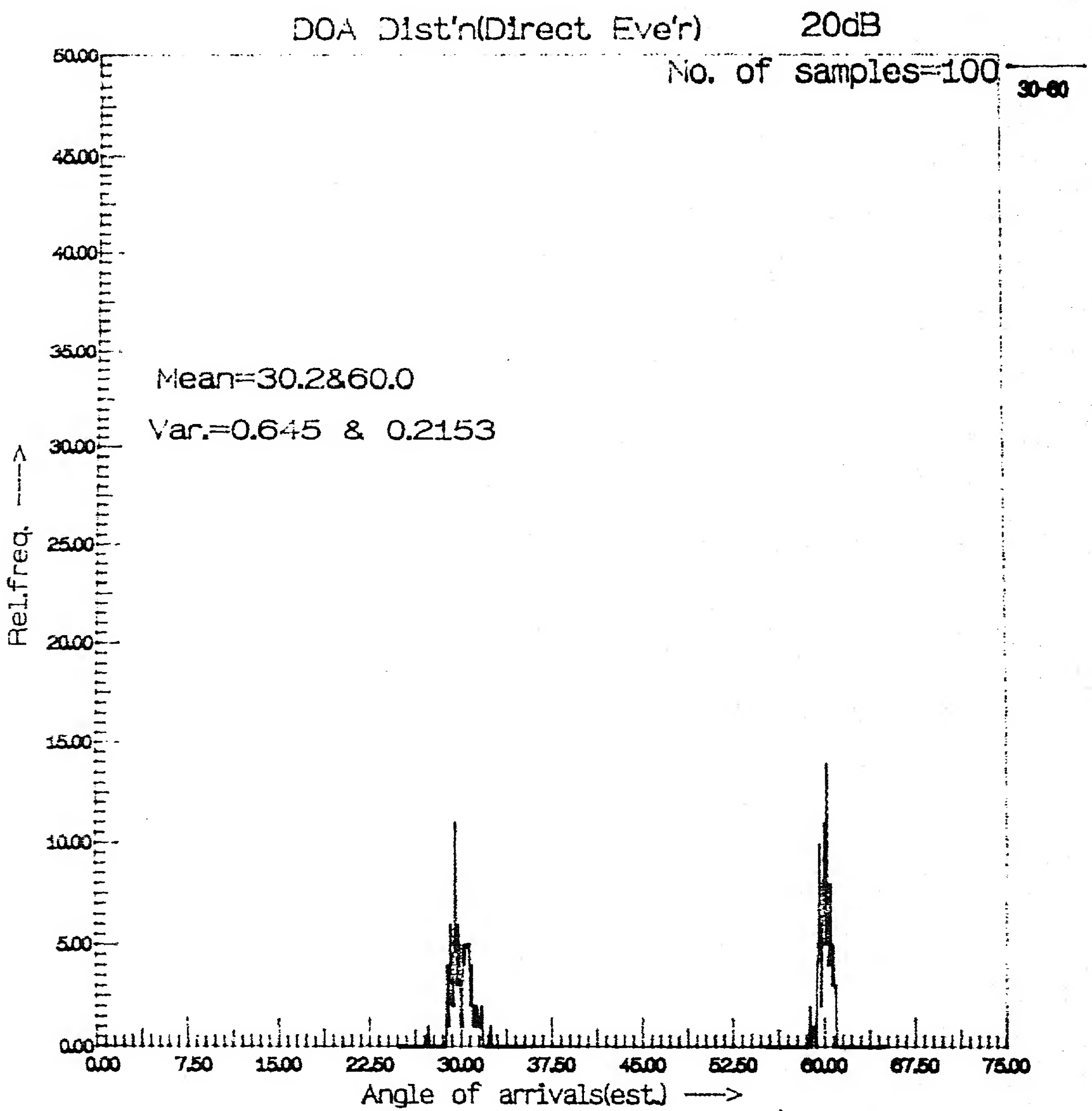

DOA Dist'n(Direct Eve'r) 20dB
No. of samples=100
30-60
Mean=30.2&60.0
Var.=0.645 & 0.2153
Rel.freq. --->
Angle of arrivals(est.) --->
0.00
5.00
10.00
15.00
20.00
25.00
30.00
35.00
40.00
45.00
50.00
7.50
22.50
37.50
45.00
52.50
60.00
67.50
75.00

Fig. 4.7

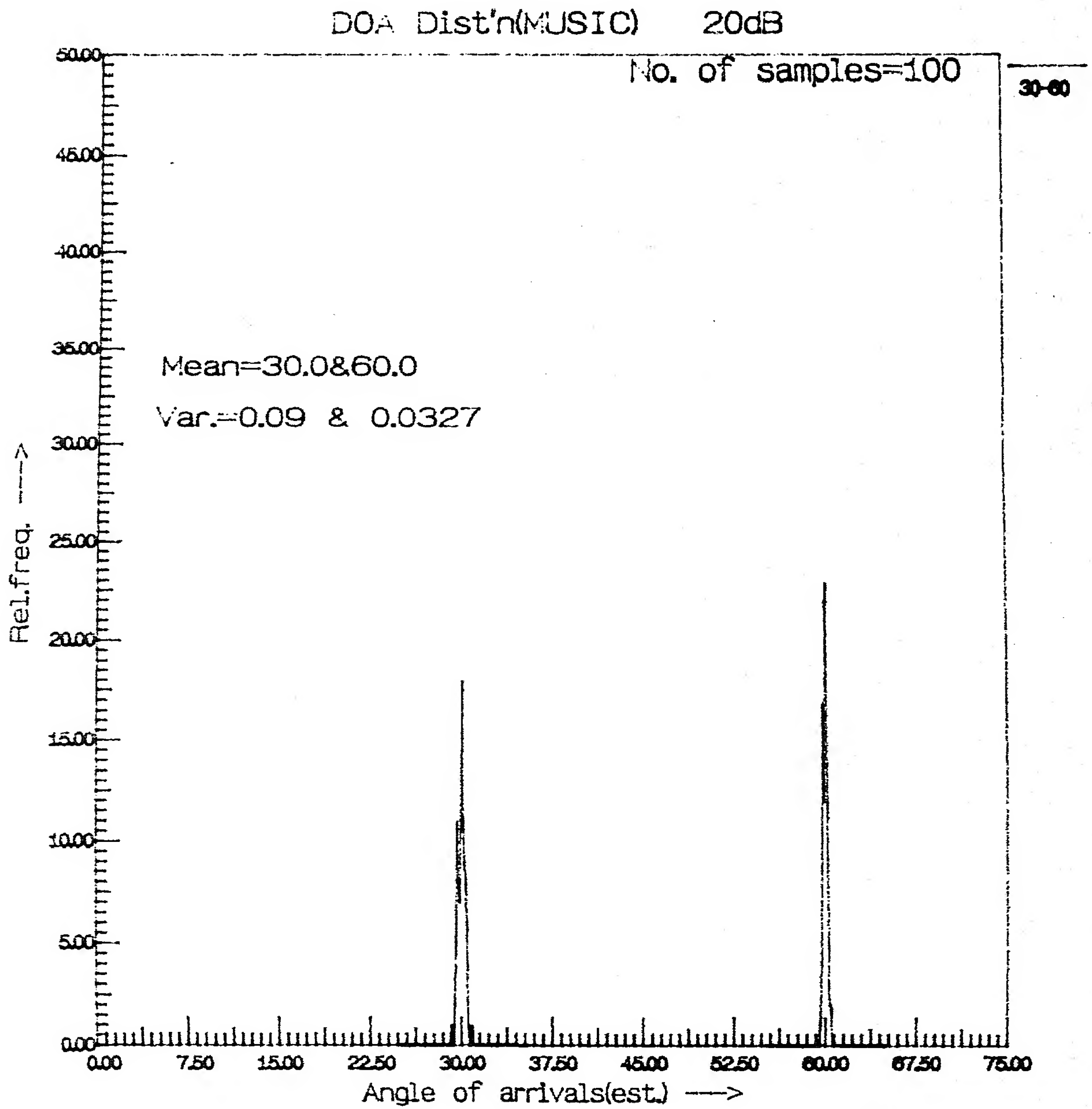
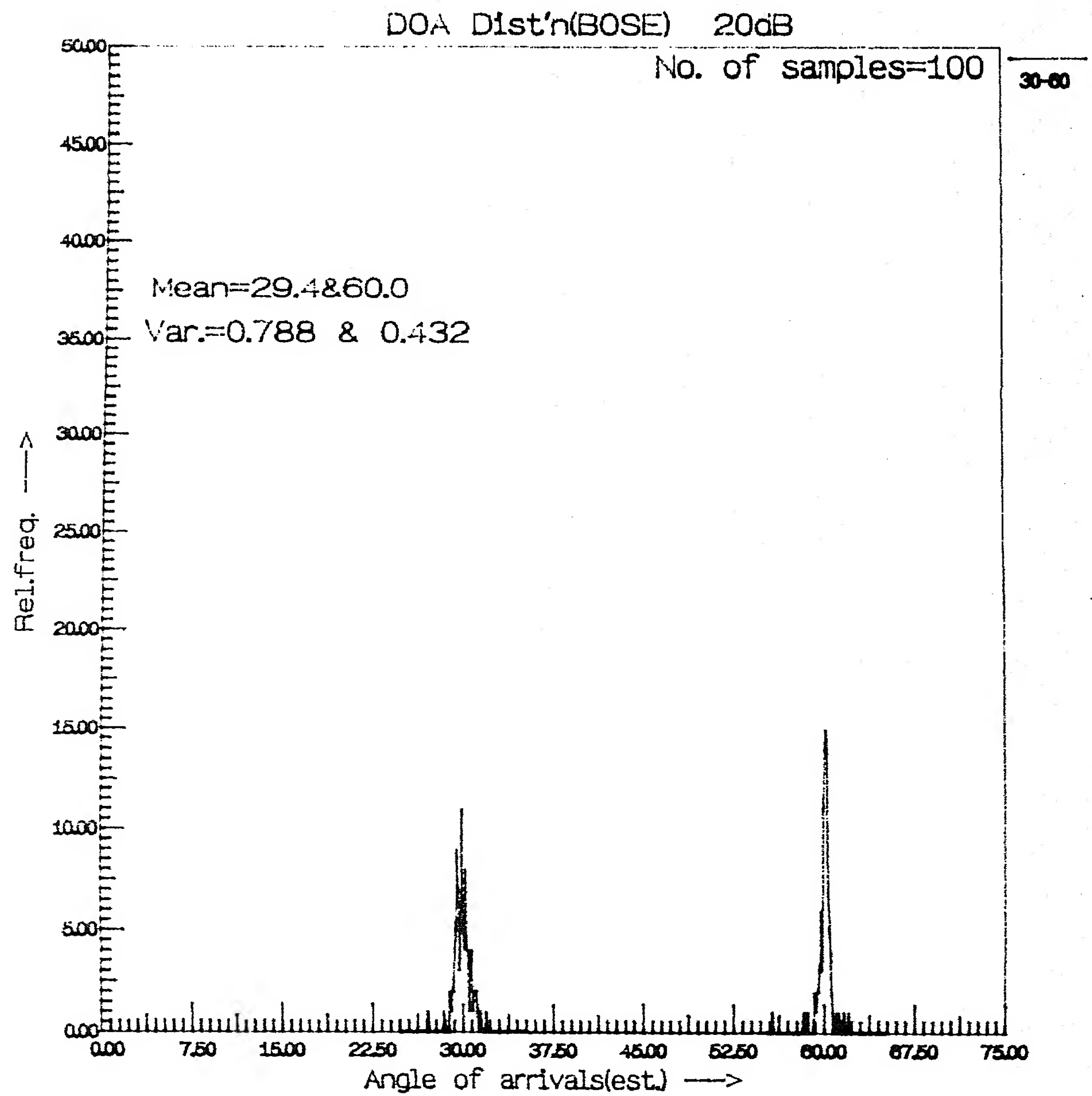

DOA Dist'n(MUSIC) 20dB
No. of samples=100
30-60
Mean=30.0&60.0
Var.=0.09 & 0.0327
Rel.freq. --->
Angle of arrivals(est.) --->
0.00
5.00
10.00
15.00
20.00
25.00
30.00
35.00
40.00
45.00
50.00
7.50
22.50
37.50
52.50
60.00
67.50
75.00

Fig. 4.8


DOA Dist'n(BOSE) 20dB
No. of samples=100
30-60
Mean=29.4&60.0
Var.=0.788 & 0.432
Rel.freq. --->
Angle of arrivals(est.) --->
0.00
5.00
10.00
15.00
20.00
25.00
30.00
35.00
40.00
45.00
50.00
7.50
15.00
22.50
30.00
37.50
45.00
52.50
60.00
67.50
75.00
Fig. 4.9

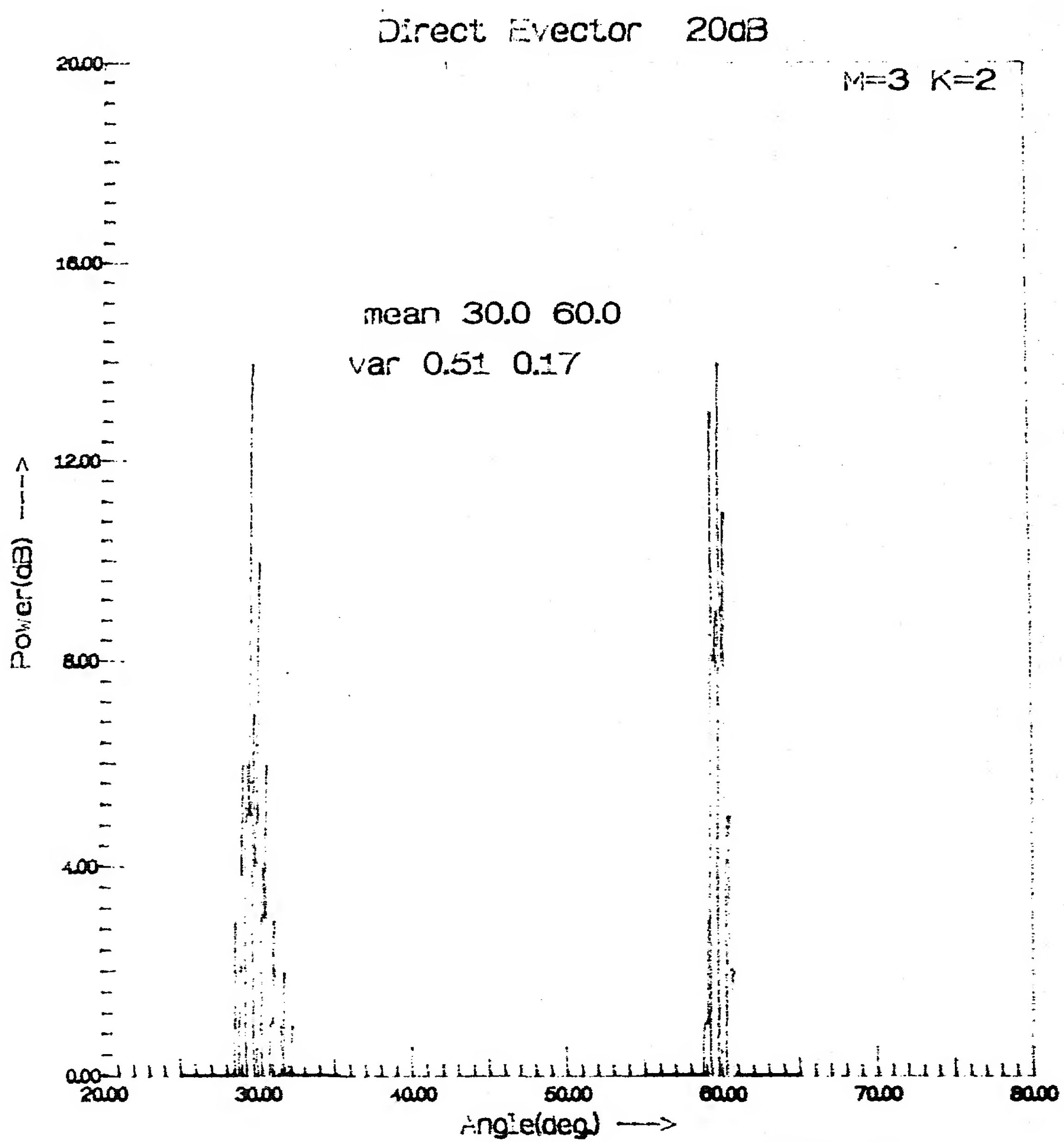

Direct Evector 20dB
M=3 K=2
mean 30.0 60.0
var 0.51 0.17
Power(dB) --->
20.00
16.00
12.00
8.00
4.00
0.00
20.00
30.00
40.00
50.00
60.00
70.00
80.00
Angle(deg) --->

Fig. 4.10

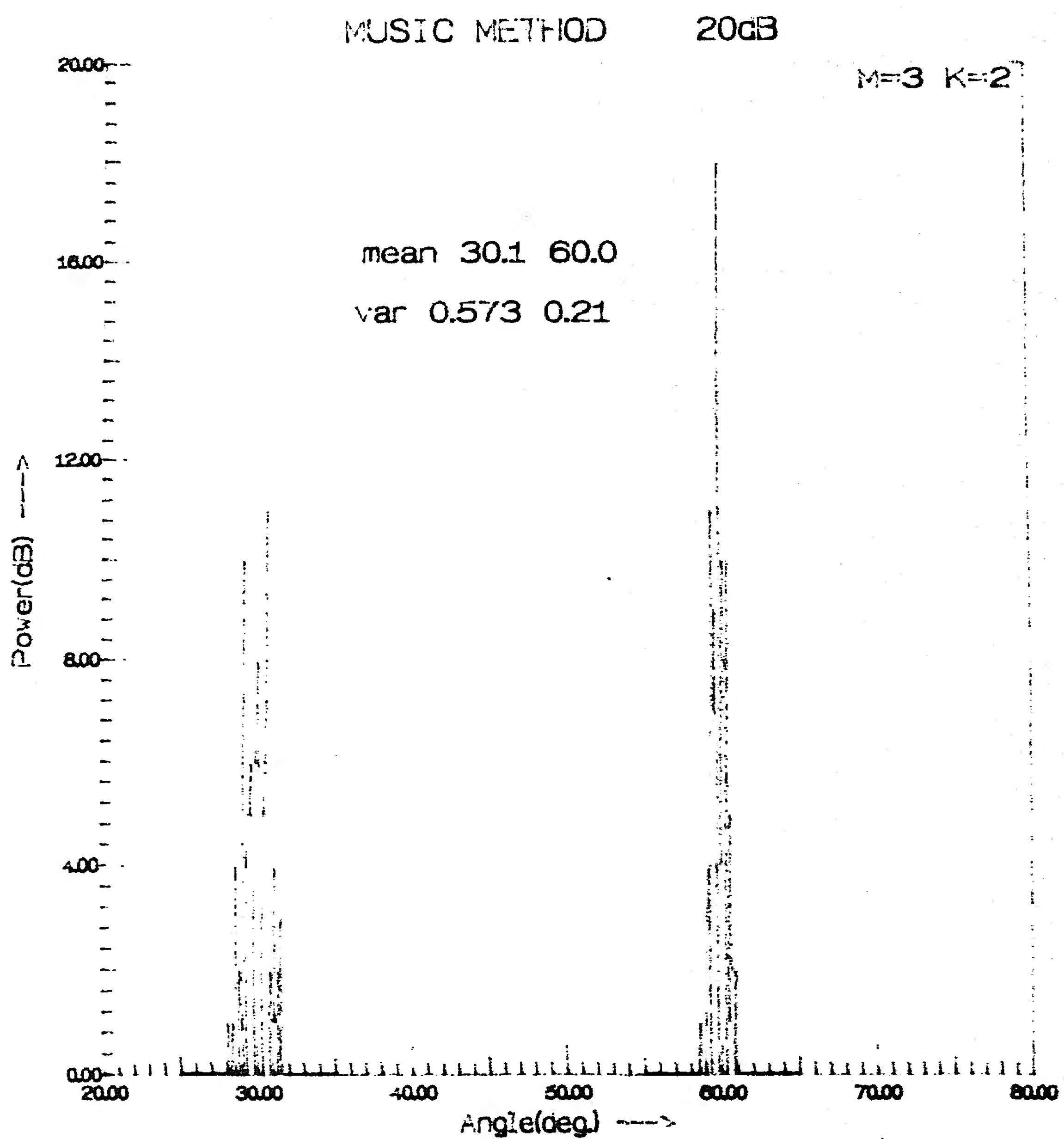

MUSIC METHOD 20dB
M=3 K=2
mean 30.1 60.0
var 0.573 0.21
Power(dB) --->
Angle(deg.) --->
0.00
4.00
8.00
12.00
16.00
20.00
20.00
30.00
40.00
50.00
60.00
70.00
80.00

Fig. 4.11

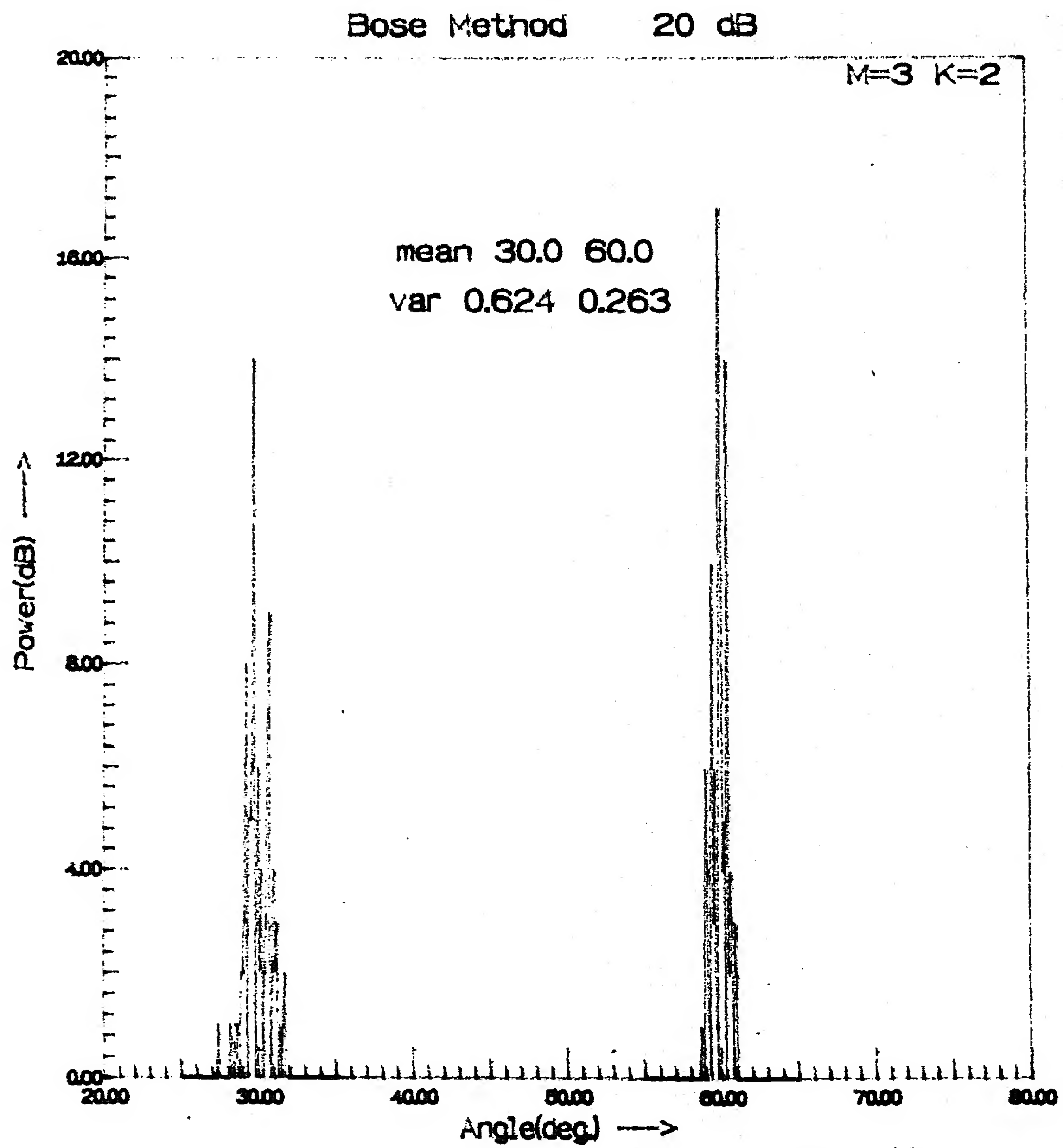
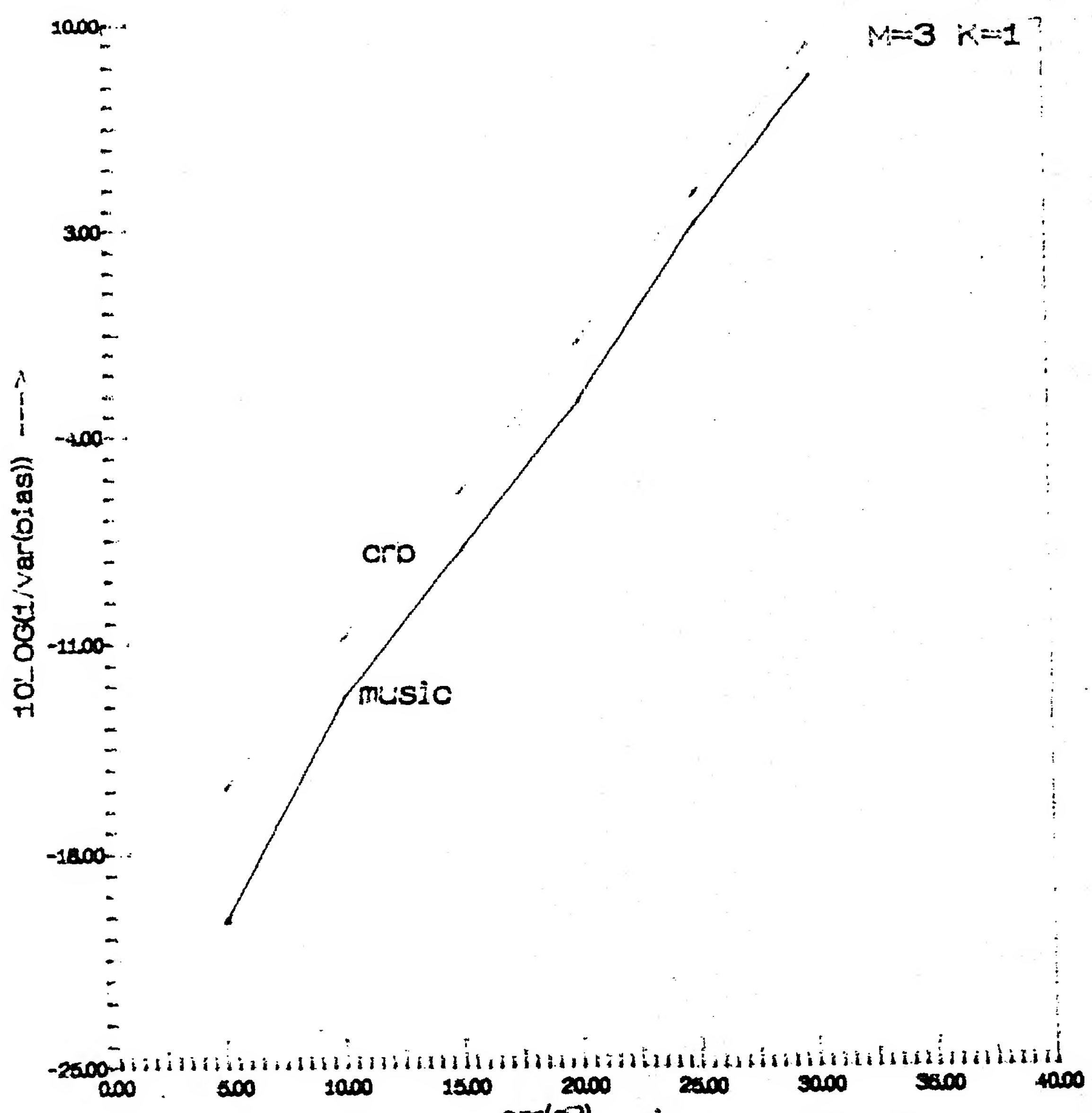

Bose Method 20 dB
M=3 K=2
mean 30.0 60.0
var 0.624 0.263
Power(dB) --->
0.00
4.00
8.00
12.00
16.00
20.00
20.00
30.00
40.00
50.00
60.00
70.00
80.00
Angle(deg.) --->

Fig. 4.12



M=3 K=1
crb
music
10LOG(1/var(bias)) --->
10.00
3.00
-4.00
-11.00
-18.00
-25.00
0.00
5.00
10.00
15.00
20.00
25.00
30.00
35.00
40.00
snr(dB) --->

Fig. 4.13

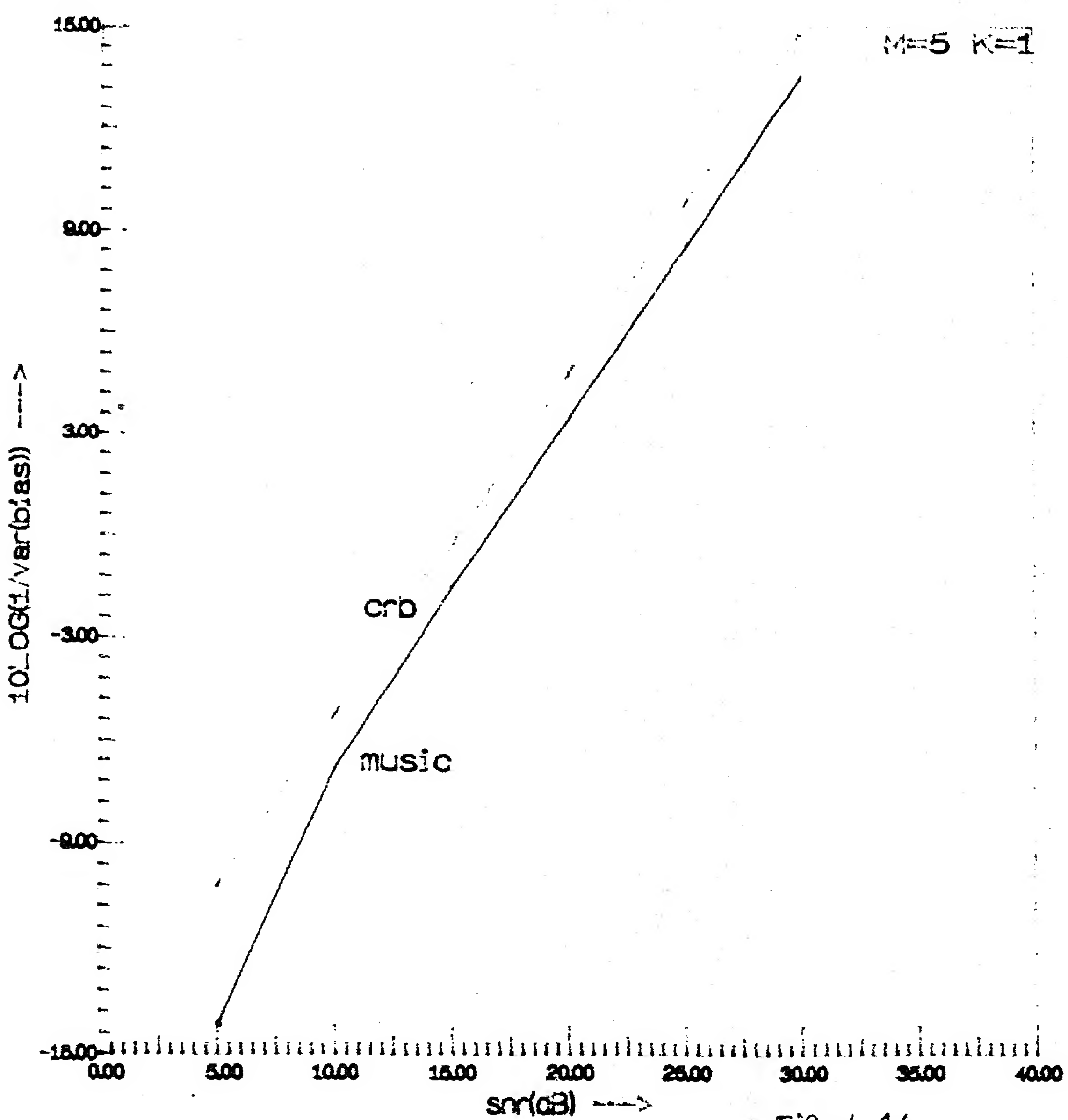

M=5 K=1
15.00
9.00
3.00
-3.00
-9.00
-15.00
10'LOG(1/var(bias)) --->
crb
music
0.00
5.00
10.00
15.00
20.00
25.00
30.00
35.00
40.00
snr(dB) --->

Fig. 4.14